Computing $1/N^2$ corrections in AdS/CFT

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ABSTRACT

Stringy corrections in AdS/CFT generally fall into the category of either α' effects or string loop effects, corresponding to $1/\lambda$ and $1/N$ corrections, respectively, in the dual field theory. While $\alpha'^3 R^4$ corrections have been well studied, at least in the context of $\mathcal{N} = 4$ super-Yang-Mills, less is known about the $1/N^2$ corrections arising from closed string loops. In this paper, we consider $\mathrm{AdS}_5 \times \mathrm{SE}_5$ compactifications of the IIB string, and compute the closed string loop correction to the anomaly coefficients a and c in the dual field theory. For $T^{1,1}$ reductions, we find the string loop correction to yield $c - a = 1/24$, which is the contribution to $c - a$ of a free $\mathcal{N} = 2$ hypermultiplet. We also comment on reductions to lower dimensional AdS theories as well as the nature of T-duality with higher derivatives.

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1 Introduction

While many important features of string theory may be investigated in its low energy limit, it is often desirable to go beyond supergravity and to examine distinguishing features that separate string theory from ordinary supergravity. Such stringy effects include Kaluza-Klein modes and non-perturbative states such as D-branes as well as string worldsheet effects arising from the string loop expansion and the α' expansion. In fact, the latter α' expansion, which is equivalent to a higher derivative expansion in the effective field theory, has attracted much recent attention for multiple reasons.

From a quantum gravity point of view, higher derivative corrections serves as a means of probing string theory at a fundamental level. This has been successfully applied to the study of stringy black holes and higher derivative effects on black hole entropy [1–4] (see *e.g.* [5,6] and references therein). Alternatively, higher derivative corrections also play an important role in AdS/CFT, showing up as finite coupling and in some cases $1/N$ effects in the dual field theory.

The string α' expansion naturally leads to a higher derivative expansion in the effective field theory, with each factor of α' accompanied by two additional derivatives. In most cases, the focus has been on the gravitation action, including $\alpha' R^2$ terms in the heterotic effective action and $\alpha'^3 R^4$ terms in the type II theories. However, the complete α' expansion involves not just Riemann terms but all fields of the theory. Such corrections in their entirety are rather complicated, and often only incomplete information is known. Nevertheless, the first higher order curvature terms are generally well established, and for non-flux backgrounds, they are often sufficient for most purposes.

The perturbative $\alpha'^3 R^4$ corrections in type II string theory arise at both tree level and one-loop order. In the IIA case, the corrections take the schematic form

$$S_{\mathrm{IIA}}[\alpha'^3] = \frac{\alpha'^3}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[e^{-2\phi}(t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10})R^4 + (t_8 t_8 - \tfrac{1}{8}\epsilon_{10}\epsilon_{10})R^4 + B \wedge t_8 R^4 \right]. \tag{1.1}$$

The $t_8 t_8 R^4$ terms were first obtained by direct calculation of four graviton scattering at tree level [7] and one loop [8] order, while the $\epsilon_{10}\epsilon_{10}R^4$ terms are related to the eight-dimensional Euler density, and first arise at the level of five graviton scattering. The one-loop CP-odd term $B \wedge X_8$ is related to the five-brane anomaly, and computed in [9,10]. This is the IIA analog of the heterotic Green-Schwarz term, and can be computed directly through a parity violating five-point amplitude [11,12] or more abstractly through the elliptic genus [13].

Because of its topological nature, the $B \wedge X_8$ term in (1.1) provides a useful handle on the study of higher derivative corrections in dimensionally reduced IIA theories. While there are often technical difficulties involved for practical calculations, abstractly once the corrections arising from $B \wedge X_8$ are pinned down, many of the remain terms may be obtained by supersymmetry. As an example of this, consider the lift of IIA to eleven-dimensional supergravity. In this case, $B \wedge X_8$ lifts to $C_3 \wedge X_8$, so that [10]

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int \left[R * 1 - \tfrac{1}{2}F_4 \wedge *F_4 - \tfrac{1}{6}C_3 \wedge F_4 \wedge F_4 + (4\pi\kappa_{11}^2)^{2/3}C_3 \wedge X_8 + \cdots \right], \tag{1.2}$$

where

$$X_8 = \frac{1}{(2\pi)^4}\left(-\frac{1}{768}(\mathrm{Tr}\, R^2)^2 + \frac{1}{192}\mathrm{Tr}\, R^4 \right). \tag{1.3}$$

Compactifying to five dimensions on a Calabi-Yau three-fold gives $\mathcal{N} = 2$ supergravity coupled to n_v vector multiplets and n_h hypermultiplets with $n_v = h_{(1,1)} - 1$ and $n_h = h_{(2,1)} + 1$ [14].

Focusing only on the vector multiplets, the compactification of (1.2) on CY_3 proceeds by expanding the Kähler form J in a basis of $(1,1)$ forms ω_I on CY_3. The Chern-Simons term then reduces in a

straightforward manner

$$\int_{\mathcal{M}_{11}} C_3 \wedge F_4 \wedge F_4 = \int_{\mathcal{M}_5} c_{IJK} A^I \wedge F^J \wedge F^K, \tag{1.4}$$

where c_{IJK} are the triple intersection numbers. Similarly, the gravitational Chern-Simons term reduces as [15, 16]

$$\int_{\mathcal{M}_{11}} C_3 \wedge X_8 = -\int_{\mathcal{M}_5} \frac{c_{2I}}{24} A^I \wedge \operatorname{Tr} R^2, \tag{1.5}$$

where c_{2I} arises from the expansion of the second Chern class

$$c_{2I} = \frac{1}{16(2\pi)^2} \int_{CY_3} \omega_I \wedge \operatorname{Tr} R^2. \tag{1.6}$$

The power of supersymmetry then enables us to deduce the entire five-dimensional $\mathcal{N} = 2$ action for the vector multiplets in terms of the topological data c_{IJK} and c_{2I}. In particular, at the R^2 level, the supersymmetric completion of $A^I \wedge \operatorname{Tr} R^2$ was obtained in [17] using superconformal tensor calculus and an off-shell formalism [18–21]. The resulting bosonic action has the form [17]

$$
\begin{aligned}
S_5 \;=\; & \frac{1}{2\kappa_5^2} \int \Big[R * 1 - \tfrac{1}{2}\mathcal{N}_{IJ} dM^I \wedge *dM^J - \tfrac{1}{2}G_{IJ}F^I \wedge *F^J - \tfrac{1}{6}c_{IJK}A^I \wedge F^J \wedge F^K \\
& - \frac{c_{2I}}{24}\big(\tfrac{1}{4}A^I \wedge \operatorname{Tr} R^2 - \tfrac{1}{8}M^I C_{\mu\nu\rho\sigma}^2 * 1 + \cdots \big) \Big].
\end{aligned}
\tag{1.7}
$$

The addition of these R^2 terms lead to corrections to the entropy of five-dimensional $\mathcal{N} = 2$ black holes [22–24]. Furthermore, as the R^2 terms are related to the five-brane anomaly, many of these entropy results are in fact exact [2, 3, 25].

1.1 R^2 corrections and AdS/CFT

The supersymmetry analysis of [17] suggests that any R^2 correction to five-dimensional $\mathcal{N} = 2$ supergravity has the form (1.7), with a precise relation between the coefficient of $C_{\mu\nu\rho\sigma}^2$ and the gravitational Chern-Simons term $A^I \wedge \operatorname{Tr} R^2$. Truncating to the pure supergravity sector and integrating out the auxiliary fields of the off-shell theory, the effective four-derivative action has the form [26]

$$
\begin{aligned}
S_5 \;=\; & \frac{1}{2\kappa_5^2} \int \Big[R * 1 - \frac{3}{2}F \wedge *F + \frac{12}{L^2} * 1 + \Big(1 - 4\frac{\alpha}{L^2} \Big) A \wedge F \wedge F \\
& - \alpha \Big(\frac{1}{4}A \wedge \operatorname{Tr} R^2 - \frac{1}{8}C_{\mu\nu\rho\sigma}^2 * 1 + \cdots \Big) \Big].
\end{aligned}
\tag{1.8}
$$

At this level, the theory is completely determined by two parameters: L, the AdS radius and α, the coefficient of the four-derivative correction terms. Note that we have chosen a non-canonical normalization for the graviphoton which however is natural in the context of IIB supergravity reduced on a Sasaki-Einstein manifold.

As highlighted above, for Calabi-Yau compactifications of eleven-dimensional supergravity, α is given by the second Chern class of CY_3. However, in an AdS/CFT setup, α also has a direct relation to the central charges of the dual gauge theory. This is perhaps best seen through the holographic Weyl anomaly [27], where the $\alpha C_{\mu\nu\rho\sigma}^2$ term in (1.8) shifts the leading supergravity result [28–31]. Anomaly matching then yields the AdS/CFT connection [26, 32]

$$\frac{L^3}{\kappa_5^2} = \frac{a}{\pi^2}, \qquad \frac{\alpha}{L^2} = \frac{c - a}{a}, \tag{1.9}$$

2

where L is the AdS radius, and where a and c are the central charges of the dual $\mathcal{N} = 1$ gauge theory.

Large N theories with an AdS dual have leading behavior $a = c \sim N^2/4$ [27]. However, from (1.9) we see that $1/N$ and further subleading corrections will show up in the dual gravity theory as R^2 corrections parametrized by α. These R^2 corrections have received much recent attention in computations of the shear viscosity and consequences for the conjectured KSS bound $\eta/s \geq 1/4\pi$ for the ratio of the shear viscosity to the entropy density of the dual gauge theory plasma [33, 34]. In particular, at linearized order, the ratio takes the form [26, 32, 35–37]

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 - \frac{\alpha}{L^2} + \cdots\right) = \frac{1}{4\pi}\left(1 - \frac{c-a}{a} + \cdots\right), \tag{1.10}$$

so that theories with $c > a$ will violate the KSS bound.

While many examples of super-Yang Mills theories are known with $c \neq a$, we are mainly interested in theories admitting a dual string description. Several explicit examples have been constructed with, *e.g.*, seven-branes and orientifolds, where $c - a \sim \mathcal{O}(N)$ [29, 38–41]. From the stringy point of view, the correction α in (1.8) arises from the effective theory of the branes at the singularities, and can be viewed as an open string effect that gives rise to a $1/N$ correction to the leading N^2 behavior of the central charges.

In this paper, we wish to examine the closed string and hence $\mathcal{O}(1)$ corrections to $c - a$ by appropriate reduction of the higher derivative terms in the bulk effective action. Focusing on Sasaki-Einstein compactifications of the IIB string, we immediately run into a puzzle. Namely, as discussed above, the α correction term in (1.8) is easily related to the reduction of the $C_3 \wedge X_8$ term in eleven-dimensional supergravity or the corresponding $B_2 \wedge X_8$ term in IIA theory. However, it is well known that such a $B_2 \wedge X_8$ term is absent in the IIB case, as the (p, q) IIB fivebrane is non-chiral. Thus, in contrast with (1.1), the α'^3 corrections in IIB have the schematic form

$$S_{\text{IIB}}[\alpha'^3] = \frac{\alpha'^3}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[e^{-2\phi}(t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10})R^4 + (t_8 t_8 + \tfrac{1}{8}\epsilon_{10}\epsilon_{10})R^4 \right], \tag{1.11}$$

where in fact the tree and loop terms combine with non-perturbative corrections into a modular-invariant form in terms of the IIB axi-dilaton. When reduced on SE_5, it is clear that the above action will not yield a non-vanishing $A \wedge \text{Tr}\, R^2$ term of the form given in (1.8). Hence this suggests that $\alpha = 0$, and therefore that all gauge theories dual to IIB string theory on $\text{AdS}_5 \times \text{SE}_5$ will have a vanishing $1/N^2$ correction to the central charges (but will still generically have $1/\lambda^{3/2}$ corrections arising from R^4 terms in the dual theory).

The above argument for the absence of $1/N^2$ corrections, however, fails to fully take stringy effects into account. In particular, as we demonstrate below, the finite volume of SE_5 leads to a non-vanishing contribution to α which is not present in uncompactified IIB theory. One way to see this is to note that any Sasaki-Einstein manifold admits a preferred U(1) fibration over a four-dimensional Kähler-Einstein base B

$$ds^2(\text{SE}_5) = ds^2(B) + (d\psi + \mathcal{A})^2, \tag{1.12}$$

where $d\mathcal{A} = 2J$, with J the Kähler form on B. This isometry circle then allows us to relate IIB theory on $\text{AdS}_5 \times \text{SE}_5$ to IIA theory on $\text{AdS}_5 \times B \times S^1$ via T-duality [42, 43]. This circle furthermore allows us to reduce first from ten to nine dimensions, and then from nine to five. In nine dimensions, the $B_2 \wedge X_8$ term in IIA theory reduces to $A_1 \wedge X_8$, where $A_\mu = B_{\mu 9}$. Under T-duality, we thus see that the compactified IIB theory necessarily has a similar term, however this time with $A_\mu = g_{\mu 9}$. Independent of the duality frame, this term reduces to five dimensions on the base B to give rise to a generically non-vanishing $1/N^2$ correction parameterized by α in (1.8).

In fact, working at finite circle radius, we demonstrate that there are a large class of gravitational and mixed Chern-Simons terms of the form $A \wedge X_8$ which arise in string theory. While some of these have been identified previously, the full story appears to be as yet incomplete. We thus begin in Section 2 with a reexamination of such terms which arise from the one-loop CP-odd sector of type II string theory, paying particular attention to the requirements of T-duality invariance. Following this, in Section 3 we compute the $\mathcal{O}(1)$ corrections to $c - a$ arising from the closed string sector of IIB theory on $\mathrm{AdS}_5 \times \mathrm{SE}_5$. Since these one-loop CP-odd terms are generic in string theory, we conclude in Section 4 with a discussion of similar corrections in AdS_4 and AdS_3 theories. We also comment on T-duality invariance in the presence of higher derivative terms in an Appendix.

2 One-loop CP-odd terms in string theory

Before considering the reduction to five dimensions, we review the origin of the $B \wedge X_8$ term in ten dimensions. The structure of this CP-odd term can be obtained from an explicit one-loop five-point computation following the procedure outlined in [11,12]. In the RNS formalism for the type II string, this parity violating term arises as a sum of two contributions, one from the odd-even and the other from the even-odd spin structure sector. Focusing on the odd-even sector, the one-loop amplitude may be set up with one vertex operator in the $(-1, 0)$ picture and the remaining four in the $(0, 0)$ picture

$$
\begin{aligned}
V^{(-1,0)}(k_0, \zeta^{(0)}) &= \zeta^{(0)}_{\mu\nu} \delta(\gamma) \psi^\mu (i\overline{\partial} X^\nu + \tfrac{1}{2}\alpha' k_0 \cdot \bar{\psi}\bar{\psi}^\nu) e^{ik_0 \cdot X}, \\
V^{(0,0)}(k_i, \zeta^{(i)}) &= \zeta^{(i)}_{\mu\nu} (i\partial X^\mu + \tfrac{1}{2}\alpha' k_i \cdot \psi\psi^\mu)(i\overline{\partial} X^\nu + \tfrac{1}{2}\alpha' k_i \cdot \bar{\psi}\bar{\psi}^\nu) e^{ik_i \cdot X},
\end{aligned}
\tag{2.1}
$$

along with a picture changing operator $\delta(\beta)\psi \cdot \partial X$ in the left-moving sector. It is conventional to take the first vertex to be the antisymmetric tensor $B_{\mu\nu}$ and the remaining four to be gravitons. However, it should be noted that the NSNS fields $h_{\mu\nu}$, $B_{\mu\nu}$ and ϕ all involve the same vertex operators, with the only difference being the nature of the polarization tensors $\zeta^{(i)}$.

The five-point function of the vertex operators (2.1) vanishes unless all ten fermion zero modes are soaked up in the odd spin structure sector. This gives rise to an amplitude

$$
\begin{aligned}
\mathcal{A} = {} & i\zeta^{(0)}_{\mu_0\nu_0} \epsilon^{\mu_0\nu_0\lambda_1\mu_1\lambda_2\mu_2\lambda_3\mu_3\lambda_4\mu_4} k^1_{\lambda_1} \cdots k^4_{\lambda_4} \zeta^{(1)}_{\mu_1\nu_1} \cdots \zeta^{(4)}_{\mu_4\nu_4} \times \\
& \int \frac{d^2\tau}{2\pi\tau_2} \frac{1}{4\tau_2} \int \frac{d^2 z_1}{2\tau_2} \cdots \int \frac{d^2 z_4}{2\tau_2} \sum_a \left\langle \prod_{i=1}^{4} (i\overline{\partial} X^{\nu_i} + \tfrac{1}{2}\alpha' k_i \cdot \bar{\psi}\bar{\psi}^{\nu_i}) e^{ik_i \cdot X} \right\rangle_a,
\end{aligned}
\tag{2.2}
$$

where the sum is over the three even spin structures. Upon integration of the vertex operators, this takes the form

$$
\mathcal{A} = i \int \frac{d^2\tau}{2\pi\tau_2} \frac{1}{4\tau_2} A(\bar{q}).
\tag{2.3}
$$

This integrand $A(\bar{q})$ is only a function of $\bar{q} = e^{-2\pi i \bar{\tau}}$, and computes the elliptic genus [13]. Proceeding either by direct computation or through the elliptic genus [9], we then see that the above amplitude gives rise to the $B_2 \wedge X_8$ term, as expected.

Before proceeding, we note that the extra factor of $1/4\tau_2$ in (2.3) arises from the zero mode contraction

$$
\langle \overline{\partial} X^\mu \partial X^\nu \rangle = -\frac{\alpha'}{8\pi\tau_2} \eta^{\mu\nu},
\tag{2.4}
$$

which we will revisit below in the compact case. This factor is crucial in showing that the amplitude is a total worldsheet derivative, so that only the $\tau_2 \to \infty$ boundary term contributes when integrating τ over the fundamental domain [9,12].

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Furthermore, as mentioned above, the closed string vertex operators (2.1) encode $h_{\mu\nu}$, $B_{\mu\nu}$ and ϕ through the polarization tensors $\zeta^{(i)}$. Thus, at this linear order, it is clear that the $B_2 \wedge X_8$ term incorporates not just R^4 but also the full set of NSNS fields according to

$$B_2 \wedge X_8(R) \quad \rightarrow \quad B_2 \wedge X_8(\hat{R}), \tag{2.5}$$

where

$$\hat{R}_{\mu\nu}{}^{\lambda\sigma} = R_{\mu\nu}{}^{\lambda\sigma} + \nabla_{[\mu}H_{\nu]}{}^{\lambda\sigma} - 2\nabla_{[\mu}\delta^\sigma_{\nu]}\nabla^\lambda\phi. \tag{2.6}$$

While this expression is linearized in H_3 and ϕ, we expect it to have a non-linear completion, so that $\hat{R}_{\mu\nu\rho\sigma}$ becomes the curvature of the connection with torsion $\hat{\omega} = \omega + H$. This completion is also needed in order to obtain a T-duality invariant combination of $g_{\mu 9}$ and $B_{\mu 9}$ under circle reduction. We discuss this point further in the Appendix.

The simple replacement of $R(\omega)$ by $\hat{R}(\hat{\omega})$, however, cannot be the entire story, as the type II string receives contributions from both the odd-even and even-odd spin structures. These contributions are essentially identical except for the important fact that $B_{\mu\nu}$, being antisymmetric, has opposite worldsheet parity from $h_{\mu\nu}$ and ϕ. Noting that the IIA and IIB amplitudes differ by a relative sign in the flip between odd-even and even-odd spin structures (because of the differing GSO projection), we finally obtain the result for the CP-odd sector

$$S_{\text{IIA}}[\alpha'^3] = \frac{(2\pi)^6\alpha'^3}{2\kappa_{10}^2} \int \left[B_2 \wedge X_8(\hat{R})\right]_{\text{odd in } B_2}, \tag{2.7}$$

and

$$S_{\text{IIB}}[\alpha'^3] = \frac{(2\pi)^6\alpha'^3}{2\kappa_{10}^2} \int \left[B_2 \wedge X_8(\hat{R})\right]_{\text{even in } B_2}. \tag{2.8}$$

In particular, the $B_2 \wedge X_8(R)$ term is projected out in the IIB case by worldsheet parity. However, terms of the form $B_2 \wedge R(\nabla H)^3$ and $B_2 \wedge R^3(\nabla H)$ survive.

2.1 Reduction to $D = 9$ and T-duality

In the above, we have demonstrated the existence of a one-loop CP-odd correction to IIB supergravity given by (2.8). However, this term by itself has no effect on the holographic $c - a$ computation, as H_3 vanishes in the $\text{AdS}_5 \times \text{SE}_5$ background. Instead, new terms will show up when IIB string theory is compactified on a circle to nine dimensions. After all, from a IIA point of view, the familiar $B_2 \wedge X_8(\hat{R})$ term in (2.7) may be reduced on a circle in the x^9 direction to give $A_1 \wedge X_8 + B_2 \wedge X_7$ where $A_\mu = B_{\mu 9}$ and X_7 is the circle reduction of X_8. Focusing on the first term, we note that T-dualizing to a IIB frame yields $A_1 \wedge X_8$ where now $A_\mu = g_{\mu 9}$. Thus T-duality guarantees that IIB theory on a circle necessarily includes a one-loop $A_1 \wedge X_8$ term. Of course, this IIB CP-odd term does not lift to ten dimensions, as it would schematically lift to $g_2 \wedge X_8$ (where $g_2 = \frac{1}{2}g_{\mu\nu}dx^\mu \wedge dx^\nu$), which however vanishes because of the symmetry of the metric.

To directly see what is happening in nine dimensions, it is instructive to return to the five-point one-loop amplitude. Assuming a circle of radius R and the first vertex in (2.1) to have a leg on the circle (so that $\zeta_{\mu\nu}^{(0)} \to \zeta_{\mu 9}^{(0)}$), we end up with a zero mode contraction

$$\langle \bar{\partial}X^9 \partial X^9 \rangle = -\left(\frac{\alpha'}{2}\right)^2 \langle p_L p_R \rangle. \tag{2.9}$$

Here

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}, \qquad p_R = \frac{n}{R} - \frac{wR}{\alpha'}, \tag{2.10}$$

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where n and w correspond to momentum and winding on the circle, and the expectation is with respect to the partition function

$$Z = \text{Tr}\, q^{(\alpha'/4)p_L^2} \bar{q}^{(\alpha'/4)p_R^2}. \tag{2.11}$$

This replaces the contraction (2.4) in the non-compact case that was used to obtain the additional factor of $1/\tau_2$ in (2.3).

To see the effect of this circle compactification, we first consider the large radius limit, $R \to \infty$. In this case, only the zero winding sector contributes, and we obtain

$$\langle \bar{\partial} X^9 \partial X^9 \rangle_{R \to \infty} = -\frac{\alpha'}{8\pi\tau_2}, \tag{2.12}$$

up to exponentially suppressed corrections. As expected, this directly reduces to the non-compact zero-mode contraction (2.4). On the other hand, in the small radius limit, $R \to 0$, only the zero momentum sector contributes, and we have instead

$$\langle \bar{\partial} X^9 \partial X^9 \rangle_{R \to 0} = \frac{\alpha'}{8\pi\tau_2}. \tag{2.13}$$

The difference in sign is apparent from the opposite sign of the winding term in p_R.

Combining the odd-even and even-odd spin structure sectors, we see that the IIA CP-odd amplitude is proportional to the factor

$$B_{\mu 9}(\langle \bar{\partial} X^9 \partial X^9 \rangle + \langle \bar{\partial} X^\mu \partial X^\mu \rangle) \sim \begin{cases} B_{\mu 9} & R \to \infty, \\ 0 & R \to 0, \end{cases} \tag{2.14}$$

while the IIB amplitude has the opposite behavior

$$g_{\mu 9}(\langle \bar{\partial} X^9 \partial X^9 \rangle - \langle \bar{\partial} X^\mu \partial X^\mu \rangle) \sim \begin{cases} 0 & R \to \infty, \\ g_{\mu 9} & R \to 0. \end{cases} \tag{2.15}$$

Since T-duality relates large and small radius compactifications of IIA and IIB theory, this result explicitly demonstrates the T-duality covariance of $B_2 \wedge X_8$ in nine dimensions with B_2 on the circle. In fact, the extension of T-duality to the full reduction of $B_2 \wedge X_8$ necessitates the use of the curvature with torsion (2.6) and the fact that $B_2 \wedge X_8(\hat{R})$ is a top form, so that T-duality will always flip between even and odd terms in B_2 in (2.7) and (2.8).

While the expressions (2.14) and (2.15) are appropriate in the large and small radii limits, we are of course interested in corrections arising at a finite radius. With the supergravity limit in mind, we work with R finite and larger than $\sqrt{\alpha'}$. As in the $R \to \infty$ limit, only the zero winding sector contributes. However, approximating the momentum sum by an integral is only valid for $\tau_2 \lesssim R^2/\alpha'$. For $\tau_2 \gtrsim R^2/\alpha'$, the zero mode contraction (2.12) becomes exponentially suppressed. Hence the radius provides a natural cutoff

$$\langle \bar{\partial} X^9 \partial X^9 \rangle_{R^2 \gg \alpha'} \sim \begin{cases} -\dfrac{\alpha'}{8\pi\tau_2} & \tau_2 \lesssim R^2/\alpha', \\ 0 & \text{otherwise}. \end{cases} \tag{2.16}$$

The implication of this is that when the zero mode contraction takes place on the x^9 circle the integral over the fundamental domain in (2.3) is cut off at $\tau_2 \sim R^2/\alpha'$. So long as $R^2 \gg \alpha'$, the boundary contribution to the amplitude is still evaluated at large τ_2 and is hence dominated by the $\bar{q}^0$ term in the elliptic genus $A(\bar{q})$. Integrating $\int d\tau_2/\tau_2^2$ up to a cutoff of R^2/α' then demonstrates that the

amplitude $\mathcal{A}$ picks up a finite radius correction factor of $1 - \alpha'/R^2$ compared to the non-compact result.

In addition, it is important to note that the discrete momentum sum for the partition function on a circle will affect the amplitude even if the zero mode contraction is in a non-compact dimension, as in (2.4). In particular, the bosonic zero mode contribution in (2.11) takes the form

$$
Z_{R^2 \gg \alpha'} \sim \begin{cases} \dfrac{1}{\sqrt{4\pi^2 \alpha' \tau_2}} & \tau_2 \lesssim R^2/\alpha', \\ \dfrac{1}{\sqrt{4\pi^2 R^2}} & \text{otherwise.} \end{cases} \tag{2.17}
$$

Since this contribution no longer falls off as $1/\sqrt{\tau_2}$ as $\tau_2 \to \infty$, the integral of the elliptic genus $A(\bar{q})$ is enhanced by a factor of $1 + \alpha'/R^2$ whenever the zero mode contraction $\langle \overline{\partial} X^\mu \partial X^\nu \rangle$ is over a non-compact dimension.

Combining these two finite radius corrections, we may now refine the above expressions (2.14) and (2.15) for the CP-odd amplitudes. In the large radius limit, the IIA amplitude is proportional to

$$
(\langle \overline{\partial} X^9 \partial X^9 \rangle + \langle \overline{\partial} X^\mu \partial X^\mu \rangle) A_1 \wedge X_8 \sim \frac{1}{2} \left[\left(1 - \frac{\alpha'}{R^2} \right) + \left(1 + \frac{\alpha'}{R^2} \right) \right] A_1 \wedge X_8 = A_1 \wedge X_8, \tag{2.18}
$$

where $A_\mu = B_{\mu 9}$, while the IIB amplitude is proportional to

$$
(\langle \overline{\partial} X^9 \partial X^9 \rangle - \langle \overline{\partial} X^\mu \partial X^\mu \rangle) A_1 \wedge X_8 \sim \frac{1}{2} \left[\left(1 - \frac{\alpha'}{R^2} \right) - \left(1 + \frac{\alpha'}{R^2} \right) \right] A_1 \wedge X_8 = -\frac{\alpha'}{R^2} A_1 \wedge X_8, \tag{2.19}
$$

where $A_\mu = g_{\mu 9}$. What this indicates is that the α'^3 corrections to the effective supergravity actions contain the following CP-odd terms in nine dimensions[1]

$$
\begin{aligned}
S_{\text{IIA}}[\alpha'^3] &= \frac{2\pi R_{\text{IIA}}}{2\kappa_{10}^2} (2\pi)^6 \alpha'^3 \int A_1 \wedge [X_8(\hat{R})]_{\text{even in } B_2}, \\
S_{\text{IIB}}[\alpha'^3] &= \frac{2\pi R_{\text{IIB}}}{2\kappa_{10}^2} \frac{(2\pi)^6 \alpha'^4}{R_{\text{IIB}}^2} \int A_1 \wedge [X_8(\hat{R})]_{\text{even in } B_2},
\end{aligned} \tag{2.20}
$$

where of course the one-form potentials correspond to $B_{\mu 9}$ and $g_{\mu 9}$ for IIA and IIB, respectively. Here we have explicitly written out the nine-dimensional Newton's constant, and furthermore these expressions are valid in the corresponding large radii limits. (For completeness, we note that there are other CP-odd term as well, such as those associated with $B_2 \wedge X_7$. However, they vanish for the $\text{AdS}_5 \times \text{SE}_5$ reduction and hence will not contribute to the $c - a$ computation.)

It is perhaps worth mentioning that the nine-dimensional CP-odd terms in (2.20) are related by naïve T-duality where $R_{\text{IIB}} = \alpha'/R_{\text{IIA}}$. Thus we could have immediately written down the $A_1 \wedge X_8$ term for IIB theory compactified on a circle based on the existence of the corresponding well-known term in IIA theory. However, T-duality takes a large radius IIB theory into a corresponding small radius IIA limit, in which case the IIA supergravity reduction is not necessarily to be trusted so that a full string calculation is warranted. Nevertheless, it is reassuring to see that the string amplitude calculation and the supergravity reduction are in perfect agreement.

---

[1]In principle, additional lower derivative terms such as $A_1 \wedge \text{Tr} R^2 \wedge F^2$ could show up at finite circle radius. These terms, however, arise from non-zero momentum or winding sectors (depending on the T-duality frame), and hence are exponentially suppressed in the compactification radius.

3 Actual computation for $\mathrm{AdS}_5 \times \mathrm{SE}_5$

We now proceed to examine the $\mathcal{O}(1)$ contribution to $c - a$ for $\mathcal{N} = 1$ theories dual to IIB theory on $\mathrm{AdS}_5 \times \mathrm{SE}_5$. Our strategy is to take advantage of the fact that any Sasaki-Einstein metric may be given in terms of a U(1) fibration over a Kähler-Einstein base B. This allows us to proceed in two steps: first reduce to nine dimensions on S^1, and then further reduce down to five dimensions on the base B. By taking the intermediate step of working in nine dimensions, we may then straightforwardly evaluate the nine-dimensional CP-odd term (2.20) to obtain the $\mathcal{O}(1)$ contribution to $c - a$.

At the two-derivative level, the full non-linear reduction of the bosonic sector of IIB theory on SE_5 was carried out in [44]. The dimensionally reduced fields $(g_{\mu\nu}, A_\mu)$ comprise the bosonic components of the $\mathcal{N} = 2$ supergraviton multiplet in five dimensions, and are related to the ten-dimensional fields according to

$$
\begin{aligned}
ds_{10}^2 &= g_{\mu\nu}dx^\mu dx^\nu + L^2 \left[ds^2(B) + (d\psi + \mathcal{A} + L^{-1}A_\mu dx^\mu)^2 \right], \\
F_5 &= (1 + *_{10})G_5, \qquad G_5 = \frac{4}{L}\epsilon_5 - L^2 J \wedge *_5 F_2, \qquad F_2 = dA_1.
\end{aligned}
\tag{3.1}
$$

Here we have written the Sasaki-Einstein metric as a U(1) bundle over B

$$
ds^2(\mathrm{SE}_5) = ds^2(B) + (d\psi + \mathcal{A})^2, \qquad d\mathcal{A} = 2J.
\tag{3.2}
$$

The resulting five-dimensional action is that of gauged $\mathcal{N} = 2$ supergravity

$$
S_5 = \frac{1}{2\kappa_5^2} \int \left[R * 1 + \frac{12}{L^2} * 1 - \frac{3}{2}F_2 \wedge *F_2 + A_1 \wedge F_2 \wedge F_2 \right],
\tag{3.3}
$$

where

$$
\frac{1}{2\kappa_5^2} = \frac{L^5 \mathrm{vol}(\mathrm{SE}_5)}{2g_s^2 \kappa_{10}^2}.
\tag{3.4}
$$

Here $\mathrm{vol}(\mathrm{SE}_5)$ is the dimensionless volume of SE_5 and $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$.

The ansatz (3.1) corresponds to the reduction of IIB theory on a circle of constant radius L. From a nine-dimensional point of view, the fields are

$$
\begin{aligned}
ds_9^2 &= g_{\mu\nu}dx^\mu dx^\nu + L^2 ds^2(B), \\
\tilde{F}_2 &= 2LJ + F_2, \\
F_4 &= 4L^3 \epsilon_4(B) - L^2 J \wedge F_2,
\end{aligned}
\tag{3.5}
$$

where $\tilde{F}_2$ is the field strength of the Kaluza-Klein gauge field $g_{\mu 9}$. This allows us to directly compute the one-loop CP-odd term (2.20) which arises after circle compactification of IIB theory. Since H_3 vanishes for this background, we may take the familiar X_8 given in (1.3). Noting that the irreducible $\mathrm{Tr}\, R^4$ term does not contribute for the direct product nine-dimensional metric (3.5), we obtain

$$
S_5[\alpha'^3] = - \left[\frac{1}{384} \frac{2\pi R}{2\kappa_{10}^2} \frac{(2\pi)^2 \alpha'^4}{R^2} \int_B \mathrm{Tr}\, R^2 \right] \int A_1 \wedge \mathrm{Tr}\, R^2.
\tag{3.6}
$$

Comparing this expression with the general four-derivative action (1.8), and using the five-dimensional Newton's constant relation (3.4) allows us to extract the effective four-derivative coefficient

$$
\alpha = \frac{g_s^2}{96} \frac{2\pi R}{L^5 \mathrm{vol}(\mathrm{SE}_5)} \frac{(2\pi)^2 \alpha'^4}{R^2} \int_B \mathrm{Tr}\, R^2
\tag{3.7}
$$

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We now use the AdS/CFT relation $4\pi g_s N = L^4/\alpha'^2$ (which is appropriate for the dual quiver gauge theories arising from a stack of D3-branes at the tip of the cone over SE$_5$) and the holographic anomaly matching relations (1.9) to write

$$a = \frac{N^2}{4} \frac{\text{vol}(\text{SE}_5)}{\pi^3}, \qquad c - a = \frac{1}{96} \frac{2\pi}{\text{vol}(S^1)} \int_B \frac{1}{8\pi^2} \text{Tr}\, R^2, \tag{3.8}$$

where $\text{vol}(S^1)$ is the dimensionless volume of the U(1) circle. The a anomaly expression is familiar[2], while the $c - a$ difference picks up a calculable contribution from the closed string sector.

Before proceeding, it is important to keep in mind that $\text{Tr}\, R^2$ in the expression for $c - a$ is composed out of the pullback of the ten-dimensional curvature onto the base B. In particular, since the compactification manifold is a fibered space, the curvature of the U(1) bundle $d\mathcal{A} = 2J$ will contribute as well the curvature of the base B. In other words, the ten-dimensional $\text{Tr}\, R^2$ will reduce to the nine-dimensional $\text{Tr}\, \tilde{R}^2$ plus terms involving the Kaluza-Klein field $g_{\mu 9}$ through its field-strength $\tilde{F}_2$. If it were not for the latter terms, then we would simply obtain

$$c - a = \frac{1}{96} \frac{2\pi}{\text{vol}(S^1)} \int_B \frac{1}{8\pi^2} \text{Tr}\, \tilde{R}^2 = -\frac{1}{96} \frac{2\pi}{\text{vol}(S^1)} \int_B p_1 = -\frac{\sigma(B)}{32} \frac{2\pi}{\text{vol}(S^1)}, \tag{3.9}$$

where $\sigma(B) = \int_B p_1/3$ is the signature of the base B. This, however, is not the complete story, as the Kaluza-Klein gauge field is non-trivial on the base as well.

3.1 The Kaluza-Klein reduction of $\text{Tr}\, R^2$

In order to evaluate the contribution of the Kaluza-Klein gauge field to $\text{Tr}\, R^2$, we take the explicit circle reduction

$$ds_{10}^2 = e^\alpha e^\alpha + e^{2\varphi}(d\psi + \tilde{A})^2. \tag{3.10}$$

The resulting spin connections are

$$\begin{aligned}
\omega^{\alpha\beta} &= \tilde{\omega}^{\alpha\beta} - \tfrac{1}{2}e^\varphi \tilde{F}^{\alpha\beta} e^9, \\
\omega^{\alpha 9} &= -\tfrac{1}{2}e^\varphi \tilde{F}^\alpha{}_\beta e^\beta - \partial^\alpha \varphi e^9,
\end{aligned} \tag{3.11}$$

where $\tilde{\omega}^{\alpha\beta}$ is the nine-dimensional spin connection computed from the nine-dimensional metric $ds_9^2 = e^\alpha e^\alpha$. The curvature two-forms are then

$$\begin{aligned}
R^{\alpha\beta} &= [\tilde{R}^{\alpha\beta} - \tfrac{1}{4}e^{2\varphi}(\tilde{F}^{\alpha\beta}\tilde{F}_{\gamma\delta} + \tilde{F}^\alpha{}_\gamma \tilde{F}^\beta{}_\delta)e^\gamma e^\delta] \\
&\quad - \tfrac{1}{2}e^\varphi[\nabla_\gamma \tilde{F}^{\alpha\beta} + 2\tilde{F}^{\alpha\beta}\partial_\gamma\varphi + \tilde{F}^\alpha{}_\gamma \partial^\beta \varphi - \tilde{F}^\beta{}_\gamma \partial^\alpha \varphi]e^\gamma e^9, \\
R^{\alpha 9} &= -\tfrac{1}{2}e^\varphi[\nabla_\beta \tilde{F}^\alpha{}_\gamma + \tilde{F}^\alpha{}_\gamma \partial_\beta \varphi + \tilde{F}^\beta{}_\gamma \partial_\alpha \varphi]e^\beta e^\gamma \\
&\quad + [\tfrac{1}{4}e^{2\varphi}\tilde{F}^{\alpha\gamma}\tilde{F}_{\beta\gamma} - \nabla^\alpha\nabla_\beta\varphi - \partial^\alpha\varphi\partial_\beta\varphi]e^\beta e^9.
\end{aligned} \tag{3.12}$$

As indicated in (3.5), for the U(1) fibered SE$_5$, the nine-dimensional graviphoton field strength is a sum of two terms: the Kähler form of B and the five-dimensional graviphoton. Since we are primarily interested in extracting the coefficient of $A \wedge \text{Tr}\, R^2$ in five dimensions, we ignore the graviphoton contribution. In this case, $\tilde{F}_2 = 2J$ is covariantly constant on B. (Note that we have explicitly scaled

---

[2]Note that we do not focus here on a possible overall $\mathcal{O}(1)$ shift $N^2 \to N^2 - 1$ that is expected to show up in the expression for a and that has been computed through quantum corrections arising from the Kaluza-Klein tower [45,46].

out the AdS radius factor L.) Furthermore, the U(1) circle has constant radius, so we set $\varphi = 0$. The curvature two-forms then simplify as

$$R^{AB} = \begin{pmatrix} \tilde{R}^{\alpha\beta} & 0 & 0 \\ 0 & \tilde{R}^{ab} - (J^{ab}J_{cd} + J^a{}_c J^b{}_d)e^c e^d & e^a e^9 \\ 0 & -e^a e^9 & 0 \end{pmatrix}. \tag{3.13}$$

Here we have further split the tangent space indices as $\alpha, \beta = 0, \ldots, 4$ in five-dimensions, $a, b = 5, \ldots, 8$ on the base B and 9 for the U(1) fiber.

A simple computation now demonstrates that

$$\text{Tr}\, R^2 = [\text{Tr}\, \tilde{R}^2]_{\text{AdS}_5} + [\text{Tr}\, \tilde{R}^2 + 4\tilde{R}^{ab}J_{ab}J + 2\tilde{R}^{ab}J_{ac}J_{bd}e^c e^d - 24J \wedge J]_B. \tag{3.14}$$

Since B is a Kähler manifold, we may use the identities $\tilde{R}^{ab}J_{ac}J_{bd} = \tilde{R}_{cd}$ and $\tilde{R}^{ab}J_{ab} = 2\rho$ (where ρ is the Ricci form) to simplify the second term above. In this case we have

$$\text{Tr}\, R^2 = [\text{Tr}\, \tilde{R}^2]_{\text{AdS}_5} + [\text{Tr}\, \tilde{R}^2 + 8\rho \wedge J - 24J \wedge J]_B. \tag{3.15}$$

Since we have scaled out the radius L, what remains is a 'unit radius' Sasaki-Einstein five-fold, where the Ricci curvature is given by $[R_{ab}]_{\text{SE}_5} = 4\delta_{ab}$. This five-dimensional Einstein condition then requires the curvature of the Kähler-Einstein base to satisfy $[\tilde{R}_{ab}]_B = 6\delta_{ab}$, so that the Ricci-form is given by $\rho = 6J$. This gives the final expression

$$\text{Tr}\, R^2 = [\text{Tr}\, \tilde{R}^2]_{\text{AdS}_5} + [\text{Tr}\, \tilde{R}^2 + 24J \wedge J]_B. \tag{3.16}$$

Note that $J \wedge J$ is twice the volume form on the base.

We now return to the expression (3.8) for $c - a$ and substitute in (3.16) for the reduction of $\text{Tr}\, R^2$ on the base B. The result is

$$\begin{aligned} c - a &= \frac{1}{96} \frac{2\pi}{\text{vol}(S^1)} \int_B \frac{1}{8\pi^2} \left(\text{Tr}\, \tilde{R}^2 + 24J \wedge J \right) \\ &= \frac{1}{32} \frac{2\pi}{\text{vol}(S^1)} \left(-\sigma(B) + \frac{2\,\text{vol}(B)}{\pi^2} \right), \end{aligned} \tag{3.17}$$

where $\text{vol}(B)$ is the dimensionless volume of the base B.

3.2 Reduction on S^5

Given the above result, we now present a few examples where $c - a$ may be directly computed. We start with IIB theory on $\text{AdS}_5 \times S^5$, which yields the familiar duality to $\mathcal{N} = 4$ super-Yang-Mills theory. In this case, the five-sphere may be written as $U(1)$ bundled over CP_2. The signature of CP_2 is $\sigma(CP_2) = 1$, while its volume is $\pi^2/2$. As a result, the two terms cancel in (3.17), and we verify that $c = a$ in $\mathcal{N} = 4$ super-Yang-Mills. Alternatively, we may compute the Riemann curvature from the Fubini-Study metric and directly show that $\text{Tr}\, \tilde{R}^2 = -24J^2$. This demonstrates that it is not just the integrated expression in (3.17), but also the integrand itself that vanishes everywhere on CP_2.

The result that $c - a$ remains unshifted by string loop contributions is of course consistent with the expectation that while decoupling of the center of mass U(1) takes U(N) to SU(N), this shift affects both a and c identically. Thus c remains identified with a, with $c = a = (N^2 - 1)/4$. Note that, since the integrand of (3.17) is trivial, this string loop correction will continue to vanish for orbifolds of S^5, such as $S^5/\mathbb{Z}_3$. Since this model is dual to $\mathcal{N} = 1$, $\text{SU}(N)^3$ gauge theory, it suggests that the $\mathcal{O}(1)$ contribution to $c - a$ ought to be $c - a = 3/16$, corresponding to three decoupled $\mathcal{N} = 1$ vectors

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from the U(1) factors. Although the string loop calculation does not give such a shift, this could be accounted for by a second contribution to $c - a$ arising from quantum corrections from the states in the Kaluza-Klein tower [45, 46]. In particular, the Kaluza-Klein modes that need to be considered are those arising from the massless fields in ten dimensions compactified on $S^5/\mathbb{Z}_3$, and it is precisely these modes that have not been captured by the string loop calculation. While the contribution of complete $\mathcal{N} = 8$ supergravity multiplets would not shift $c - a$, the orbifolding breaks this to $\mathcal{N} = 2$, in which case their contributions would no longer be expected to vanish. It of course remains to be seen whether a direct computation along the lines of [45, 46] will reproduce the predicted value of $c - a = 3/16$.

3.3 Reduction on $T^{1,1}$

The next simplest case to consider is IIB theory on $\mathrm{AdS}_5 \times T^{1,1}$ [47]. Since the base of $T^{1,1}$ is $S^2 \times S^2$, we simply have $\mathrm{Tr}\,\tilde{R}^2 = 0$. (The vanishing of the signature can also be understood from the existence of both a self-dual and an anti-self dual harmonic two-form.) In this case we are left with the volume term $\mathrm{vol}(B) = (4\pi/6)^2 = 4\pi^2/9$. Substituting this into (3.17) then gives the $T^{1,1}$ result

$$c - a = \frac{1}{24}, \tag{3.18}$$

where we also used the fact that the volume of the U(1) fiber is $4\pi/3$. Curiously, this is the contribution to $c - a$ for a free $\mathcal{N} = 2$ hypermultiplet or negative that of a vector multiplet[3].

It would be interesting to see how this result may arise from the perspective of the dual gauge theory. Generalizing the idea of the $S^5/\mathbb{Z}_3$ orbifold, we focus on $\mathcal{N} = 1$, SU(N) quiver gauge theories. Since we expect $c - a$ to count the number of decoupled $\mathcal{N} = 1$ vectors, the natural prediction would be

$$c - a = \frac{(\#\ \text{of nodes in the quiver})}{16}. \tag{3.19}$$

This gives $c - a = 1/8$ for the conifold gauge theory, which however does not agree with (3.18). While it is possible that we have lost a factor of three in the string loop computation, we instead suggest as above that there is a second contribution to $c - a$ from the Kaluza-Klein tower so that

$$c - a = \frac{1}{24}\bigg|_{\text{string loop}} + \frac{1}{12}\bigg|_{\text{supergravity loop}} = \frac{1}{8}, \tag{3.20}$$

where the supergravity loop contribution arises from the Kaluza-Klein tower in five dimensions. While it would be interesting to perform such a calculation, in practice the non-trivial Kaluza-Klein spectroscopy on $T^{1,1}$ [48] would appear to make this a challenge.

3.4 Reduction on $Y^{p,q}$

We now turn to reductions on the Sasaki-Einstein manifold $Y^{p,q}$, which are dual to a large family of $\mathcal{N} = 1$ superconformal quiver gauge theories [49–51]. On the IIB supergravity side, the Sasaki-Einstein manifold $Y^{p,q}$ has topology $S^2 \times S^3$. The metric in canonical form (1.12) is given by

$$
\begin{aligned}
ds^2 &= \frac{1 - cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dy^2}{w(y)q(y)} + \frac{w(y)q(y)}{36}(d\beta + c\cos\theta d\phi)^2 \\
&\quad + \frac{1}{9}[d\psi' - \cos\theta d\phi + y(d\beta + c\cos\theta d\phi)]^2,
\end{aligned}
\tag{3.21}
$$

---

[3]Although we believe the sign of the correction to be correct, keeping track of the sign conventions for the CP-odd terms is rather subtle. In principle, a reliable means of fixing the sign may be through the reduction of R^4 in the CP-even sector, as the sign of the Weyl-squared contribution in (1.8) is unambiguous.

where

$$w(y) = \frac{2(a - y^2)}{1 - cy}, \qquad q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2}. \tag{3.22}$$

The parameters a and c are to be chosen to avoid conical singularities at the poles $y_1 \leq y \leq y_2$ where y_1 and y_2 are the two smallest roots of the cubic $a - 3y^2 + 2cy^3 = 0$. Note that $c = 0$ corresponds to $T^{1,1}$, while if $c \neq 0$ we can rescale the coordinates to set $c = 1$.

In order to compute $c - a$ given by (3.17), we take the natural vielbein basis

$$e_1 = \sqrt{\frac{1 - cy}{6}} d\theta, \qquad e_2 = \sqrt{\frac{1 - cy}{6}} \sin\theta d\phi,$$

$$e_3 = \frac{1}{\sqrt{w(y)q(y)}} dy, \qquad e_4 = \frac{\sqrt{w(y)q(y)}}{6}(d\beta + c\cos\theta d\phi). \tag{3.23}$$

For the U(1) fibration, we have

$$\mathcal{A} = -\frac{1}{3}[\cos\theta d\phi - y(d\beta + c\cos\theta d\phi)], \tag{3.24}$$

and it is easy to verify that $d\mathcal{A} = 2J$ where $J = e_1 \wedge e_2 + e_3 \wedge e_4$.

Given the above metric, we may directly compute $\mathrm{Tr}\, \tilde{R}^2$ on the base

$$\mathrm{Tr}\, \tilde{R}^2 = 24 \left[\frac{(1 - ac^2)^2}{(1 - cy)^6} - 1 \right] J \wedge J, \tag{3.25}$$

where

$$\frac{1}{2} J \wedge J = \frac{1 - cy}{36} \sin\theta d\theta \wedge d\phi \wedge dy \wedge d\beta \tag{3.26}$$

is the volume form on the base. In order to proceed, we need to integrate $\mathrm{Tr}\, \tilde{R}^2 + 24 J \wedge J$ on the base B. However, a difficulty arises in that for generic values of p and q the manifold $Y^{p,q}$ is irregular. This means that while the above expressions are valid locally the base B is ill defined as a base manifold. At best, for appropriate values of p and q the Sasaki-Einstein space is quasi-regular, and the base is then an orbifold.

Although the base B may be ill defined, the five-dimensional Sasaki-Einstein manifold itself is smooth and free of curvature singularities. Thus instead of computing the signature and volume of B separately, we directly evaluate the five-dimensional quantity

$$\mathrm{Tr}\, R^2 = \mathrm{Tr}\, \tilde{R}^2 + 24 J \wedge J = 24 \frac{(1 - ac^2)^2}{(1 - cy)^6} J \wedge J. \tag{3.27}$$

Integrating this over the entire SE$_5$ then gives

$$\int_{\mathrm{SE}_5} \mathrm{Tr}\, R^2 \wedge \tfrac{1}{3} d\psi' = \frac{64\pi^3}{3} \ell(y_2 - y_1)(1 + c(y_2 + y_1)), \tag{3.28}$$

where $\ell = P_1/p = P_2/q$ is related to the period of the U(1) fiber in the notation of [49]. As a result, we find

$$c - a = \frac{1}{96} \frac{2\pi}{\mathrm{vol}(S^1)^2} \int_{\mathrm{SE}_5} \frac{1}{8\pi^2} \mathrm{Tr}\, R^2 \wedge \tfrac{1}{3} d\psi' = \frac{1}{8} \left(\frac{2\pi/3}{\mathrm{vol}(S^1)} \right)^2 \ell(y_2 - y_1)(1 + c(y_2 + y_1)). \tag{3.29}$$

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Relating ℓ and the roots y_1 and y_2 to the Chern numbers p and q finally gives

$$c - a = \frac{1}{48} \left(\frac{2\pi/3}{\mathrm{vol}(S^1)} \right)^2 \frac{p(4p^2 - 9q^2) + (2p^2 + 3q^2)\sqrt{4p^2 - 3q^2}}{p^2(p^2 - q^2)}. \tag{3.30}$$

For p and q chosen appropriately, the square root becomes rational, and the base B is an orbifold. In this case, orbits of the U(1) fiber close, and we may take $\mathrm{vol}(S^1) = 2\pi/3$, corresponding to 2π periodicity of the ψ' circle. However, for irregular $Y^{p,q}$ the orbits do not close, and this suggests that $\mathrm{vol}(S^1)$ should be taken to be infinite, in which case $c - a$ would vanish. This difference in behavior for quasi-regular versus irregular Sasaki-Einstein manifolds appears rather unusual, and merits further investigation.

Based on the dual quiver gauge theory with $2p$ gauge groups, we may expect from (3.19) that $c - a = p/8$. If this is the case, then the additional contribution from the Kaluza-Klein tower would have to compensate for the rather unwieldy function of p and q appearing in (3.30).

4   Graviphoton backgrounds and lower dimensional AdS reductions

While we have mainly focused on AdS$_5$ reductions of IIB supergravity, similar features arise when examining AdS$_4$ reductions of eleven dimensional supergravity on Sasaki-Einstein seven-folds given by a non-trivial circle fibration over a six-dimensional Kähler-Einstein base. In particular, we demonstrate that the reduction of $C_3 \wedge X_8$ gives rise to four-dimensional couplings of the $\mathcal{N} = 2$ graviphoton T with the curvature tensor of the form

$$R_{\mu\nu\lambda\sigma} T^{\mu\nu} T^{\lambda\sigma}, \tag{4.1}$$

and further compute its coefficient.

So far, we had not been concerned with nontrivial ten-dimensional graviphoton backgrounds, since they do not give rise to AdS$_5$ compactifications. However, for AdS$_4$ reductions of eleven-dimensional supergravity on SE$_7$, the ten-dimensional graviphoton is important, and some field redefinitions may be required in order to write down a consistent (one-loop) ten-dimensional action. To see how this arises, we first look at the reduction of $C_3 \wedge X_8$ to ten dimensions on a non-trivially fibered circle.

It is convenient to work on a twelve-dimensional manifold Y_{12} whose boundary is the eleven-dimensional spacetime $X_{11} = \partial Y_{11}$. We are interested in the case where X_{11} is a circle fibration over ten-dimensional spacetime M_{10}: $U(1) \to X \xrightarrow{\pi} M$. (In turn, M is a boundary to an eleven-dimensional manifold Y_{11}). The isometry is generated by a vector field v and the dual global connection one form is denoted by e:

$$\imath_v e = 1, \qquad d\,e = \pi^* T, \tag{4.2}$$

where T is the graviphoton field strength.

Now consider the circle reduction:

$$\int_X C_3 \wedge X_8(TX) = \int_{Y_{12}} G_4 \wedge X_8 \longrightarrow \int_{Y_{11}} \imath_v[G_4 \wedge \tilde{X}_8], \tag{4.3}$$

where the tilde eight-form is a polynomial of ten (rather than eleven)-dimensional curvatures and the graviphoton T. Every eleven-dimensional quantity respects the isometry, *i.e.* has a vanishing Lie derivative with respect to the vector v: $\mathcal{L}_v(.) = (d\imath_v + \imath_v d)(.) = 0$. This means in particular that closed forms upon reduction yield closed forms of lesser rank.

For now let us ignore the sources and take $dG_4 = 0$. Then $\mathcal{L}_v G = 0$ allows to write

$$G_4 = \pi^* F_4 + \pi^* H_3 \wedge e, \tag{4.4}$$

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where F_4 and H_3 are ten-dimensional RR and NSNS fluxes respectively ($\imath_v G = -H_3$). The closure of G leads to a pair of ten-dimensional equations:

$$dF_4 - H_3 \wedge T = 0\,, \qquad dH_3 = 0\,. \tag{4.5}$$

Similarly,

$$X_8 = \pi^* \tilde{X}_8 + \pi^* \tilde{X}_7 \wedge e\,, \tag{4.6}$$

where $\tilde{X}_7 = -\imath_v X_8$. Note that all quantities with tildes are polynomials in the (ten-dimensional) curvature R and the graviphoton T: $\tilde{X}_n = \tilde{X}_n(\hat{R}^{11}) = \tilde{X}_n(R^{10}, T)$. The closure of X_8 leads to:

$$d\tilde{X}_8 - \tilde{X}_7 \wedge T = 0\,, \qquad d\tilde{X}_7 = 0\,. \tag{4.7}$$

Moreover locally $X_8 = dX_7^{(0)}$. If not only X_8 but also the descendant $X_7^{(0)}$ respects the isometry, i.e. $\mathcal{L}_v X_7^{(0)}$, it follows that

$$\imath_v X_8 = -d(\imath_v X_7^{(0)})\,.$$

One can show $\tilde{X}_7 = -\imath_v X_8$ is not only closed, but is also exact: $\tilde{X}_7 = d\tilde{X}_6(R, T)$. Note that the horizontal eight-form $\tilde{X}_8 - T \wedge \tilde{X}_6$ is closed. We shall give explicit expressions for the polynomials with tildes shortly.

For now, let us get back to the reduction of the one-loop term (4.3):

$$
\begin{aligned}
\int_{Y_{12}} G_4 \wedge X \quad \longrightarrow \quad & \int_{Y_{11}} F_4 \wedge \tilde{X}_7 \ + H_3 \wedge \tilde{X}_8 \\
= & \int_{Y_{11}} d(F_4 \wedge \tilde{X}_6) - dF_4 \wedge \tilde{X}_6 + H_3 \wedge \tilde{X}_8 \\
= & \int_{Y_{11}} d(F_4 \wedge \tilde{X}_6) + H_3 \wedge [\tilde{X}_8 - T \wedge \tilde{X}_6] \\
= & \int_{M_{10}} F_4 \wedge \tilde{X}_6 \ + B_2 \wedge [\tilde{X}_8 - T \wedge \tilde{X}_6]\,.
\end{aligned}
\tag{4.8}
$$

Note that the RR four-form appearing in (4.8) satisfies the correct Bianchi identity, $dF_4 - H_3 \wedge T = 0$, while B_2 comes wedged with a closed eight-form, hence ensuring that the one-loop term (in the absence of fivebrane sources) is invariant under the NSNS gauge transformation $\delta B_2 \to d\Lambda_1$.

While further reductions are not important for our purposes, clearly we can extend this discussion to the case of multiple (commuting) isometries of the eleven-dimensional background. Let us discuss the reduction to nine dimensions. Starting from eleven dimensions, we should consider now a pair of isometries generated by v^1 and v^2. Here we concentrate on $B_1 = \imath_{v^1} \imath_{v^2} C_3$ couplings. Even though the SL(2) doublets are not important for our purposes, and (in IIA language) we shall set the pair of graviphotons to zero, let us have a look at the complete set of lower dimensional descendants of $C_3 \wedge X_8$. Due to isometries we can write

$$G_4 = \pi^* F_4 + \pi^* F_3^i \wedge e^i + \pi^* H_2 \wedge e^1 \wedge e^2\,, \tag{4.9}$$

where $H_2 = dB_1$. One can check:

$$dF_4 - F_3^i \wedge T^i = 0\,, \qquad dF_3^i - \epsilon^{ij} H_2 T^j = 0\,, \qquad dH_2 = 0\,. \tag{4.10}$$

Similarly for X_8:

$$X_8 = \pi^* \tilde{X}_8 + \pi^* \tilde{X}_7^i \wedge e^i + \pi^* \tilde{X}_6 \wedge e^1 \wedge e^2\,, \tag{4.11}$$

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where now $\tilde{X}_n = \tilde{X}_n(\hat{R}^{11}) = \tilde{X}_n(R^9, T^1, T^2)$. The closure of X_8 leads to:

$$d\tilde{X}_8 - \tilde{X}_7^i \wedge T^i = 0, \qquad d\tilde{X}_7^i - \epsilon^{ij}\tilde{X}_6 \wedge T^j = 0, \qquad d\tilde{X}_6 = 0. \qquad (4.12)$$

Let us introduce quantities:

$$I_5 = d^{-1}\tilde{X}_6, \qquad I_6^i = d^{-1}[\tilde{X}_7^i - \epsilon^{ij}I_5 \wedge T^j] = 0, \qquad (4.13)$$

and note that due to $\epsilon_{ij}T^i \wedge T^j = 0$

$$d[\tilde{X}_8 - I_6^i \wedge T^i] = d\tilde{X}_8 - \tilde{X}_7^i \wedge T^i = 0. \qquad (4.14)$$

The result of the reduction of $C_3 \wedge X_8$ is then

$$\int_{M_9} B_1 \wedge [\tilde{X}_8 - I_6^i \wedge T^i] - F_4 \wedge I_5 + \epsilon_{ij}F_3^i \wedge I_6^i, \qquad (4.15)$$

where F_4 and F_3^i satisfy the Bianchi identities (4.10). Once again, in absence of fivebranes, the action is invariant under $\delta B_1 \to d\Lambda_0$.

From now on, we shall consider only ten-dimensional theories with a single non-trivial graviphoton. A quick comment about anomalies is in order. Indeed the term $B_2 \wedge [\tilde{X}_8 - T \wedge \tilde{X}_6]$ (just like its eleven-dimensional ancestor) is not invariant under ten-dimensional diffeomorphisms. For that matter, even the closed eight-form is not; $\tilde{X}_8 - T \wedge \tilde{X}_6$ is invariant only under the combined action of ten-dimensional diffeomorphisms and graviphoton U(1) transformations. In the presence of NS5-branes, $dH = \eta(W_6 \hookrightarrow M_{10})$, the variation will produce a complicated expression

$$d^{-1}\delta d^{-1}[\tilde{X}_8 - T \wedge \tilde{X}_6], \qquad (4.16)$$

restricted to the fivebrane worldvolume. Note that from the other side $\tilde{X}_8$ is simply a sum of the usual (closed) X_8 polynomial and a part that depends on the graviphoton, $\tilde{X}_8 = X_8(TM_{10}) + X_8(R, T)$. Hence in a trivial graviphoton background, $T = 0$, we recover the usual ten-dimensional anomaly inflow[4]. Of course, eleven-dimensional anomaly cancellation requires contributions from three sources — the fivebrane anomaly, the variations from the bulk $C_3 \wedge X_8(TM_{11})$ and the (modified) Chern-Simons term $C_3 \wedge G_4 \wedge G_4$ [52]. The prediction of this argument is that the reduction of the latter should yield counterparts to (4.16)[5].

4.1 Higher derivative couplings in AdS$_4$

Before turning to the reduction, it is instructive now to look at explicit expressions. Since the single trace part of X_8 does not contribute to the reductions, it is sufficient to look at $X_4 = p_1(TM_{11})$ and $\tilde{X}_4$ and $\tilde{X}_3 = d\tilde{X}_2$ arising from the reduction:

$$8\pi^2 \tilde{X}_4 = R^{ab} \wedge R^{ba} - (R^{ab}T^{ba}) \wedge T - \frac{1}{2}R^{ab} \wedge T^b \wedge T^a + \frac{1}{4}T^{ab}T^{ba}\,T \wedge T$$
$$+ \frac{1}{4}T^{ab}\,T^b \wedge T^a \wedge T + \frac{1}{2}\nabla^c T^a \wedge \nabla^d T^a \wedge e^c \wedge e^d,$$

---

[4]With an abuse of notation, $X(TM_D)$ refers to forms constructed of polynomials in D-dimensional curvatures. Quantities with tilde $\tilde{X}$ are polynomials in R and T.

[5]A circumstantial argument in favor of this is given by recalling that the contribution from $C_3 \wedge G_4 \wedge G_4$ is important when the normal bundle is not trivial. It is not hard to check that when the normal bundle N is trivial and $M_{10} = W_6 \times N$, the lift to eleven dimensions is provided by fibering the M-theory circle over N. In this situation the graviphoton field strength T is a horizontal form on T and hence pulls back to zero on W_6. In other words, when the normal bundle is trivial, the coupling $B_2 \wedge [(\tilde{X}_8 - X_8(TM_{10})) - T \wedge \tilde{X}_6]$ is invariant.

$$8\pi^2 \tilde{X}_3 = \left(R^{ab} - \frac{1}{2}T^{ab}\,T - \frac{1}{4}T^a \wedge T^b \right) \wedge e^c\, \nabla^c T^{ba} + \frac{1}{2}e^c \wedge \nabla^c T^a \wedge (T^{ad}\,T^d), \qquad (4.17)$$

where $T = \frac{1}{2}T_{ab}\,e^a \wedge e^b$ and $T^a = T^a{}_b\,e^b$ and the covariant derivative ∇ is taken with respect to the Levi-Civita connection. (All the curvatures here are ten-dimensional, and $a, b, c, \ldots$ are ten-dimensional tangent space indices.) One can now compute

$$\tilde{X}_2 = d^{-1}\tilde{X}_3 = \frac{1}{8\pi^2}\left(R^{ab} - \frac{1}{4}T^{ab}T - \frac{1}{4}T^a \wedge T^b \right) T^{ba}, \qquad (4.18)$$

and see that it is invariant. Hence $\tilde{X}_3$ is cohomologically trivial.

While we have reduced $C_3 \wedge X_8$ from eleven dimensions, as highlighted in (2.7), the string loop amplitude necessarily involves the curvature of the connection with torsion (2.6). In particular, we need to make the replacement $R \Rightarrow \hat{R}$ inside $\tilde{X}$ in order to account for H contributions to the NSNS part of the couplings. However, these contributions are not important in the SE$_7$ reduction since we backgrounds have vanishing H.

We are now ready to discuss the graviphoton couplings in AdS$_4$. From the original eleven-dimensional point of view, the Sasaki-Einstein reduction takes the form [53]

$$ds_{11}^2 = g_{\mu\nu}dx^\mu dx^\nu + L^2[ds^2(B) + (d\psi + \mathcal{A} + (2L)^{-1}A_\mu dx^\mu)^2],$$
$$G_4 = \frac{6}{L}\epsilon_4 - \frac{L^2}{2}J \wedge *_4 F_2, \qquad F_2 = dA_1. \qquad (4.19)$$

The resulting four-dimensional action corresponds to minimal gauged $\mathcal{N} = 2$ supergravity with AdS radius $L/2$

$$S_4 = \frac{1}{16\pi G_4}\int \left[R * 1 + \frac{24}{L^2} * 1 - \frac{1}{2}F_2 \wedge *F_2 \right]. \qquad (4.20)$$

In ten dimensions, we may view this solution as a reduction of IIA supergravity on a six-dimensional Kähler-Einstein manifold B with a Kähler form J. The Sasaki-Einstein seven-fold is then obtained by taking a U(1) bundle over B with $d\mathcal{A} = 2J$. From this point of view, the four-dimensional Newton's constant is given by

$$\frac{1}{16\pi G_4} = \frac{L^6 \text{vol}(B)}{2\kappa^2}. \qquad (4.21)$$

We are of course interested in the one-loop correction to (4.20). In the absence of the NSNS B field, the relevant term in (4.8) is $\int_{M_{10}} F_4 \wedge \tilde{X}_6$. Taking into account the constant factors in (2.7), we find

$$S_4[\alpha'^3] = \frac{g_s^2(2\pi)^6\alpha'^3}{2\kappa^2}\int_{\text{AdS}_4 \times B} F_4 \wedge \tilde{X}_6 = -\frac{1}{96}\frac{g_s^2(2\pi)^6\alpha'^3}{2\kappa^2}\int_{\text{AdS}_4 \times B} F_4 \wedge \tilde{X}_4 \wedge \tilde{X}_2. \qquad (4.22)$$

Focusing on the $\mathcal{N} = 2$ graviphoton, F_2, we pick out the component $F_4 = -(L^2/2)J \wedge *_4 F_2$ from (4.19), so that

$$\begin{aligned} S_4[\alpha'^3] &= \frac{1}{192}\frac{g_s^2(2\pi)^6 L^2\alpha'^3}{2\kappa^2}\int_{\text{AdS}_4 \times B} *_4F_2 \wedge \tilde{X}_2 \wedge J \wedge \tilde{X}_4 \\ &= \frac{1}{16\pi G_4}\frac{1}{192}\frac{g_s^2(2\pi)^6\alpha'^3}{L^4\text{vol}(B)}\int_{\text{AdS}} *_4F_2 \wedge \tilde{X}_2 \int_B J \wedge \tilde{X}_4, \qquad (4.23) \end{aligned}$$

where we have also used (4.21).

Making use of (4.18), and taking the AdS$_4$ graviphoton to be $T = F_2$ gives

$$*_4 F_2 \wedge \tilde{X}_2 = \frac{1}{8\pi^2} \frac{1}{12} [R_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} - \tfrac{1}{2} F^4 - \tfrac{1}{4}(F^2)^2] *_4 1. \tag{4.24}$$

As a result, the four-dimensional action (4.20) picks up a correction at the four derivative level

$$S_4[\alpha'^3] = \frac{1}{16\pi G_4} \int \alpha L^2 \left(R_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} - \tfrac{1}{2} F^4 - \tfrac{1}{4}(F^2)^2 + \cdots \right) * 1, \tag{4.25}$$

where the ellipsis denotes terms that we have not focused on. See [54] for a recent discussion of these terms.

The coefficient α may be extracted from (4.23)

$$\alpha = \frac{1}{192} \frac{g_s^2 (2\pi)^4 \alpha'^3}{L^6 \mathrm{vol}(B)} \frac{1}{24} \int_B J \wedge \tilde{X}_4 = \frac{1}{192} \frac{g_s^2 (2\pi)^2 \alpha'^3}{L^6 \mathrm{vol}(B)} \frac{1}{48} \int_B J \wedge (\mathrm{Tr}\, \tilde{R}^2 + 32 J \wedge J), \tag{4.26}$$

where we have made use of the six-dimensional version of (3.16). Finally, if the AdS$_4$ geometry arose from a stack of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, we may use the relation $L^6 = 32\pi^2 g_s^2 k N \alpha'^3$ to write

$$\alpha = \frac{\lambda}{N^2} \frac{1}{9 \cdot 8192 \,\mathrm{vol}(B)} \int_B J \wedge (\mathrm{Tr}\, \tilde{R}^2 + 32 J \wedge J), \tag{4.27}$$

where $\lambda = N/k$ [55]. This integral vanishes on CP_3 (the base for S^7), but is generally non-zero.

Note that in addition to (4.25), there are four-derivative order terms that are in general moduli dependent. Indeed, for $B = b^i \omega^i$ where $\omega^i \in H^2(B)$, the reduction yields

$$\int_{\mathrm{AdS}_4 \times B} b^i \omega^i \wedge \mathrm{Tr}\, R^2 \wedge (\mathrm{Tr}\, R^2 - 8\pi^2 T \wedge \tilde{X}_2) = \int_{\mathrm{AdS}_4} \alpha^i\, b^i\, \mathrm{Tr}\, R^2, \tag{4.28}$$

where $\alpha^i = \int_B \omega^i \wedge (\mathrm{Tr}\, R^2 + 8 J \wedge \rho - 32 J \wedge J) = \int_B \omega^i \wedge (\mathrm{Tr}\, \tilde{R}^2 + 64 J \wedge J)$. For $\mathcal{N} = 8$ reductions, i.e. where $B = CP_3$, the second cohomology is one-dimensional and $\omega = J$. Moreover in this case b is constant, and the resulting $\int_{\mathrm{AdS}_4} \mathrm{Tr}\, R^2$ correction is non-dynamical.

4.2   Reducing to AdS$_3$

We conclude with a brief discussion of reductions to AdS$_3$. In this case, it is convenient to pass via six dimensions. The lowest order in derivatives one-loop contributions are well-studied in the context of IIA/heterotic duality with 16 supercharges (see e.g. [9,10]), and can be collected into a Chern-Simons like term

$$B \wedge \left(\mathcal{F}^T L \mathcal{F} - \mathrm{Tr}\, R^2 \right), \tag{4.29}$$

where $\mathcal{F}^T = (T, G_2, F_2^I)$ with T as above denoting the IIA RR one-form (graviphoton), while G_2 and F_2^I descend from the RR 3-form (see [10] for details). The intersection matrix is given by $L = [\sigma^1 \oplus d_{IJ}]$, where d_{IJ} is in turn the intersection matrix of the internal space K and $I, J = 1, ..., h^2(K)$. For theories with 16 supercharges K is a $K3$ surface, and d_{IJ} has signature $(3, 19)$. We shall mostly ignore these modes and concentrate on T and G_2, since these are terms that also survive the truncation to theories with lower supersymmetry.

Our discussion makes it clear that there are two types of higher-derivative (but still one-loop) corrections to this coupling. Indeed in ten dimensions we have both new terms involving ∇H and H^2 and terms involving T. While along internal directions these vanish, they should appear in the

six-dimensional effective theory. After integration by parts, the Chern-Simons term (4.29) takes the form

$$H \wedge d^{-1}\left(d_{IJ} F^I \wedge F^J + T \wedge G_2 + 8\pi^2(\tilde{X}_4 - T \wedge \tilde{X}_2) + \cdots\right). \tag{4.30}$$

The complete coupling should have $O(4, 20)$ symmetry and hence the modified curvature terms should be written in terms of $\mathcal{F}$ and not simply T. However we do not try to impose this here, and just write down the terms that involve the graviphoton (and the ellipsis stands for the rest). The reduction to AdS$_3$ can now be readily performed, and yields a correction term of the form

$$\int_{\text{AdS}_3} d^{-1}\left(d_{IJ} F^I \wedge F^J + T \wedge G_2 + 8\pi^2(\tilde{X}_4 - T \wedge \tilde{X}_2) + \cdots\right), \tag{4.31}$$

containing higher derivative terms in in addition to the expected gauge and gravitational Chern-Simons terms.

Acknowledgments

A T-duality and higher-order terms in the effective action

We shall present here a very brief (and incomplete) discussion of the action of T-duality on the corrections to the effective action. Since the terms in the effective action built solely out of curvature cannot be invariant under T-duality, the corrections discussed in Section 2 can also be seen as completions required to make higher curvature terms invariant.

One way of introducing a correction with torsion is to consider the Courant bracket—a generalization of the Lie bracket acting on vector fields, which however acts on sections of the so called generalized tangent bundle E. The latter locally is a product of tangent and cotangent bundles to the manifold M $(\dim(M) = d)$:

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0. \tag{A.1}$$

Sections of E are called generalized vectors. Locally they can be written as $X = x + \xi$ where $x \in TM$ and $\xi \in T^*M$. In going from one coordinate patch U_α to another U_β, we have to first make the usual patching of vectors and one-forms, and then give a further patching describing how T^*M is fibered over TM in E. The choice of B gives a canonical identification of E with $T \oplus T^*$, and the ordinary Courant bracket on E gets identified with the twisted Courant bracket on $T \oplus T^*$.

The generalized connection is defined by analogy to the ordinary connection and is an operator

$$D : C^\infty(W) \to C^\infty(E \otimes W), \tag{A.2}$$

where W is some vector bundle which carries a representation of $O(d, d)$. We can now think of D as $D = \partial + \Omega$, where the ordinary derivative ∂ simply gives a term in the T^*M part of E and nothing in the TM part. Thus one defines the derivative D, acting on a generalized vector X. The generalized connection is now defined by the Courant bracket:

$$[x + \xi, y + \eta] = [x, y]_{\text{Lie}} + \mathcal{L}_x \eta - \mathcal{L}_y \xi - \frac{1}{2} d\left(\iota_x \eta - \iota_y \xi\right), \tag{A.3}$$

where $[x, y]_{\text{Lie}}$ is the usual Lie bracket between vectors and $\mathcal{L}_x$ is the Lie derivative.

We are interested in a case when the string background admits an isometry, hence both the metric on M and H are annihilated by the Lie derivative of some vector v, $\mathcal{L}_v g = \mathcal{L}_v H = 0$. We shall use the setup similar to that of Section 4. In particular we may use $H = \pi^* H_3 + (\pi^* H_2) \wedge e$, where $\iota_v e = 1$ and $de = \pi^* F$.

We may also decompose the sections of $TM \oplus T^*M$ into horizontal and vertical components, $x \longrightarrow x + fv$ and $\rho \longrightarrow \rho + \phi e$ correspondingly, and consider the Courant bracket

$$[(x + fv; \rho + \phi e), \, (y + gv; \lambda + \omega e)]_{(H_3, H_2)} =$$

$$[(x; \rho), \, (y; \lambda)]_{H_3} + \left(0 + (\mathcal{L}_x g - \mathcal{L}_y f) \, v; \xi_{\text{base}} + (\mathcal{L}_x \omega - \mathcal{L}_y \phi) e \right), \tag{A.4}$$

where the first term is the Courant bracket on the base of the circle fibration and

$$\xi_{\text{base}} = \quad (\; \iota_x F \quad \iota_x H_2 \;) \, \eta \begin{pmatrix} g \\ \omega \end{pmatrix} - (\; \iota_y F \quad \iota_y H_2 \;) \, \eta \begin{pmatrix} f \\ \phi \end{pmatrix} \tag{A.5}$$

$$+ (\; g \quad \omega \;) \, \eta \begin{pmatrix} df \\ d\phi \end{pmatrix} - \tfrac{1}{2} d \left((\; g \quad \omega \;) \, \eta \begin{pmatrix} f \\ \phi \end{pmatrix} \right),$$

with $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One can now readily check that an $O(1,1)$ transformation

$$\begin{pmatrix} F \\ H_2 \end{pmatrix} \to X \cdot \begin{pmatrix} F \\ H_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f \\ \phi \end{pmatrix} \to X \cdot \begin{pmatrix} f \\ \phi \end{pmatrix}, \quad \begin{pmatrix} g \\ \omega \end{pmatrix} \to X \cdot \begin{pmatrix} g \\ \omega \end{pmatrix} \tag{A.6}$$

leaves ξ_{base} invariant and is an automorphism of the bracket (A.4), provided that X is an $O(1,1)$ matrix $X^T \eta X = \eta$. Hence the connection $\omega + H$ defined by (A.4) is T-duality invariant. Finally, by flipping the sign of H, we may remark that the connection $\omega - H$ is T-duality anti-invariant.

This can be generalized for the case of multiple commuting isometries v_I, provided $\iota_{v_I} \iota_{v_J} H = 0$ for any two vectors v_I and v_J. In general, for n isometries, $O(n, n)$ transformations are not an automorphism of the Courant bracket, and hence one cannot construct an $O(n, n)$ invariant generalized (twisted) connection. More details on generalized connection can be found in [56–58].

One may also use the Courant bracket to define a curvature operator that will be tensorial when restricted to integrable maximally isotropic subbundles of E. For our purpose, it suffices to look at the curvature $\hat{R}$ written in (2.6) in the linearized approximation. Once more, $\hat{R}_+ = \hat{R}(\omega + H)$ is T-duality invariant, while $\hat{R}_- = \hat{R}(\omega - H)$ does not transform particularly nicely under T-duality. However writing locally $X_{4n}(\hat{R}_\pm) = dX_{4n-1}^{(0)}(\omega \pm H)$, we recall that the latter contain only odd powers of the connection, and hence $X_8(\hat{R}_+)$ and $X_8(\hat{R}_-)$ are respectively even and odd under T-duality.

With all this in mind, the CP-odd corrections (2.7), (2.8) and (2.20) can be summarized as the T-duality invariant combination

$$(\; \gamma \quad \beta \;) \, \eta \begin{pmatrix} X_8(\hat{R}_+) + X_8(\hat{R}_-) \\ X_8(\hat{R}_+) - X_8(\hat{R}_-) \end{pmatrix}, \tag{A.7}$$

where the curvature expressions are constructed out of the original ten-dimensional fields. From the IIA point of view, we have introduced $\beta = B_{\text{inv}} + e \wedge \iota_v B$, with $B_{\text{inv}} = (1 - e \wedge \iota_v) B - \tfrac{1}{2} \iota_v g \wedge \iota_v B$ being the component of the B-field invariant under T-duality. We have also introduced $\gamma = (\alpha'/R^2) e \wedge \iota_v g$ which vanishes in the absence of isometries and is suppressed in the large radius IIA ten-dimensional limit. Hence in ten dimensions the formula reproduces the known CP-odd one-loop terms. Upon reduction on a circle, (A.7) correctly reproduces the nine dimensional couplings. Note that T-duality

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exchanges the roles of $\imath_v B$ and $\imath_v g$, so that the former becomes associated with γ and the latter with β from the IIB point of view. Of course, this cannot be the complete story, as it ought to be possible to express the CP-even corrections in a T-duality invariant manner as well. However, this looks somewhat more involved, as the circle reduction of $\hat{R}_\pm$ appears rather unenlightening.

References

[1] J. M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, JHEP **9712** (1997) 002 [arXiv:hep-th/9711053].

[2] J. A. Harvey, R. Minasian and G. W. Moore, *Non-abelian tensor-multiplet anomalies*, JHEP **9809** (1998) 004 [arXiv:hep-th/9808060].

[3] P. Kraus and F. Larsen, *Microscopic Black Hole Entropy in Theories with Higher Derivatives*, JHEP **0509**, 034 (2005) [arXiv:hep-th/0506176].

[4] A. Sen, *Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity*, JHEP **0509**, 038 (2005) [arXiv:hep-th/0506177].

[5] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, Gen. Rel. Grav. **40**, 2249 (2008) [arXiv:0708.1270 [hep-th]].

[6] A. Castro, J. L. Davis, P. Kraus and F. Larsen, *String Theory Effects on Five-Dimensional Black Hole Physics*, Int. J. Mod. Phys. A **23**, 613 (2008) [arXiv:0801.1863 [hep-th]].

[7] D. J. Gross and E. Witten, *Superstring Modifications Of Einstein's Equations*, Nucl. Phys. B **277**, 1 (1986).

[8] N. Sakai and Y. Tanii, *One Loop Amplitudes And Effective Action In Superstring Theories*, Nucl. Phys. B **287**, 457 (1987).

[9] C. Vafa and E. Witten, *A One Loop Test Of String Duality*, Nucl. Phys. B **447**, 261 (1995) [arXiv:hep-th/9505053].

[10] M. J. Duff, J. T. Liu and R. Minasian, *Eleven-dimensional origin of string / string duality: A one-loop test*, Nucl. Phys. B **452**, 261 (1995) [arXiv:hep-th/9506126].

[11] W. Lerche, B. E. W. Nilsson and A. N. Schellekens, *Heterotic String Loop Calculation Of The Anomaly Cancelling Term*, Nucl. Phys. B **289**, 609 (1987).

[12] W. Lerche, B. E. W. Nilsson, A. N. Schellekens and N. P. Warner, *Anomaly cancelling terms from the elliptic genus*, Nucl. Phys. B **299**, 91 (1988).

[13] E. Witten, *Elliptic genera and quantum field theory*, Commun. Math. Phys. **109**, 525 (1987).

[14] A. C. Cadavid, A. Ceresole, R. D'Auria and S. Ferrara, *Eleven-dimensional supergravity compactified on Calabi-Yau threefolds*, Phys. Lett. B **357**, 76 (1995) [arXiv:hep-th/9506144].

[15] S. Ferrara, R. R. Khuri and R. Minasian, *M-Theory on a Calabi-Yau Manifold*, Phys. Lett. B **375**, 81 (1996) [arXiv:hep-th/9602102].

[16] I. Antoniadis, S. Ferrara, R. Minasian and K. S. Narain, R^4 *couplings in M- and type II theories on Calabi-Yau spaces*, Nucl. Phys. B **507**, 571 (1997) [arXiv:hep-th/9707013].

[17] K. Hanaki, K. Ohashi and Y. Tachikawa, *Supersymmetric Completion of an R^2 Term in Five-Dimensional Supergravity*, Prog. Theor. Phys. **117**, 533 (2007) [arXiv:hep-th/0611329].

[18] T. Kugo and K. Ohashi, *Supergravity tensor calculus in 5D from 6D*, Prog. Theor. Phys. **104**, 835 (2000) [arXiv:hep-ph/0006231].

[19] E. Bergshoeff, S. Cucu, M. Derix, T. de Wit, R. Halbersma and A. Van Proeyen, *Weyl multiplets of N = 2 conformal supergravity in five dimensions*, JHEP **0106**, 051 (2001) [arXiv:hep-th/0104113].

[20] T. Fujita and K. Ohashi, *Superconformal tensor calculus in five dimensions*, Prog. Theor. Phys. **106**, 221 (2001) [arXiv:hep-th/0104130].

[21] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *N = 2 supergravity in five dimensions revisited*, Class. Quant. Grav. **21**, 3015 (2004) [Class. Quant. Grav. **23**, 7149 (2006)] [arXiv:hep-th/0403045].

[22] A. Castro, J. L. Davis, P. Kraus and F. Larsen, *5D Attractors with Higher Derivatives*, JHEP **0704**, 091 (2007) [arXiv:hep-th/0702072].

[23] A. Castro, J. L. Davis, P. Kraus and F. Larsen, *5D Black Holes and Strings with Higher Derivatives*, JHEP **0706**, 007 (2007) [arXiv:hep-th/0703087].

[24] A. Castro, J. L. Davis, P. Kraus and F. Larsen, *Precision entropy of spinning black holes*, JHEP **0709**, 003 (2007) [arXiv:0705.1847 [hep-th]].

[25] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, JHEP **0601**, 022 (2006) [arXiv:hep-th/0508218].

[26] S. Cremonini, K. Hanaki, J. T. Liu and P. Szepietowski, *Black holes in five-dimensional gauged supergravity with higher derivatives*, JHEP **0912**, 045 (2009) [arXiv:0812.3572 [hep-th]].

[27] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].

[28] S. Nojiri and S. D. Odintsov, *On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*, Int. J. Mod. Phys. A **15**, 413 (2000) [arXiv:hep-th/9903033].

[29] M. Blau, K. S. Narain and E. Gava, *On subleading contributions to the AdS/CFT trace anomaly*, JHEP **9909**, 018 (1999) [arXiv:hep-th/9904179].

[30] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, *Diffeomorphisms and holographic anomalies*, Class. Quant. Grav. **17**, 1129 (2000) [arXiv:hep-th/9910267].

[31] M. Fukuma, S. Matsuura and T. Sakai, *Higher-derivative gravity and the AdS/CFT correspondence*, Prog. Theor. Phys. **105**, 1017 (2001) [arXiv:hep-th/0103187].

[32] A. Buchel, R. C. Myers and A. Sinha, *Beyond $\eta/s = 1/4\pi$*, JHEP **0903**, 084 (2009) [arXiv:0812.2521 [hep-th]].

[33] P. Kovtun, D. T. Son and A. O. Starinets, *Holography and hydrodynamics: Diffusion on stretched horizons*, JHEP **0310**, 064 (2003) [arXiv:hep-th/0309213].

[34] P. Kovtun, D. T. Son and A. O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231].

[35] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, *Viscosity Bound Violation in Higher Derivative Gravity*, Phys. Rev. D **77**, 126006 (2008) [arXiv:0712.0805 [hep-th]].

[36] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, *The Viscosity Bound and Causality Violation*, Phys. Rev. Lett. **100**, 191601 (2008) [arXiv:0802.3318 [hep-th]].

[37] Y. Kats and P. Petrov, *Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory*, JHEP **0901**, 044 (2009) [arXiv:0712.0743 [hep-th]].

[38] A. Fayyazuddin and M. Spalinski, *Large N superconformal gauge theories and supergravity orientifolds*, Nucl. Phys. B **535**, 219 (1998) [arXiv:hep-th/9805096].

[39] O. Aharony, A. Fayyazuddin and J. M. Maldacena, *The large N limit of $\mathcal{N} = 2, 1$ field theories from three-branes in F-theory*, JHEP **9807**, 013 (1998) [arXiv:hep-th/9806159].

[40] O. Aharony, J. Pawelczyk, S. Theisen and S. Yankielowicz, *A note on anomalies in the AdS/CFT correspondence*, Phys. Rev. D **60**, 066001 (1999) [arXiv:hep-th/9901134].

[41] O. Aharony and Y. Tachikawa, *A holographic computation of the central charges of $d = 4, \mathcal{N} = 2$ SCFTs*, JHEP **0801**, 037 (2008) [arXiv:0711.4532 [hep-th]].

[42] M. J. Duff, H. Lu and C. N. Pope, *Supersymmetry without supersymmetry*, Phys. Lett. B **409**, 136 (1997) [arXiv:hep-th/9704186].

[43] M. J. Duff, H. Lu and C. N. Pope, *$AdS_5 \times S^5$ untwisted*, Nucl. Phys. B **532**, 181 (1998) [arXiv:hep-th/9803061].

[44] A. Buchel and J. T. Liu, *Gauged supergravity from type IIB string theory on $Y^{p,q}$ manifolds*, Nucl. Phys. B **771**, 93 (2007) [arXiv:hep-th/0608002].

[45] A. Bilal and C. S. Chu, *A note on the chiral anomaly in the AdS/CFT correspondence and $1/N^2$ correction*, Nucl. Phys. B **562**, 181 (1999) [arXiv:hep-th/9907106].

[46] P. Mansfield, D. Nolland and T. Ueno, *Order $1/N^2$ test of the Maldacena conjecture. II: The full bulk one-loop contribution to the boundary Weyl anomaly*, Phys. Lett. B **565**, 207 (2003) [arXiv:hep-th/0208135].

[47] I. R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, Nucl. Phys. B **536**, 199 (1998) [arXiv:hep-th/9807080].

[48] A. Ceresole, G. Dall'Agata, R. D'Auria *et al.*, *Spectrum of type IIB supergravity on $AdS_5 \times T^{11}$: Predictions on $\mathcal{N} = 1$ SCFT's*, Phys. Rev. **D61**, 066001 (2000) [arXiv:hep-th/9905226].

[49] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$*, Adv. Theor. Math. Phys. **8**, 711 (2004) [arXiv:hep-th/0403002].

[50] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, Commun. Math. Phys. **262**, 51 (2006) [arXiv:hep-th/0411238].

[51] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, JHEP **0506**, 064 (2005) [arXiv:hep-th/0411264].

[52] D. Freed, J. A. Harvey, R. Minasian and G. W. Moore, *Gravitational anomaly cancellation for M-theory fivebranes*, Adv. Theor. Math. Phys. **2** (1998) 601 [arXiv:hep-th/9803205].

[53] J. P. Gauntlett and O. Varela, *Consistent Kaluza-Klein Reductions for General Supersymmetric AdS Solutions*, Phys. Rev. D **76**, 126007 (2007) [arXiv:0707.2315 [hep-th]].

[54] R. C. Myers, S. Sachdev and A. Singh, *Holographic Quantum Critical Transport without Self-Duality*, arXiv:1010.0443 [hep-th].

[55] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, $\mathcal{N} = 6$ *superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **0810**, 091 (2008) [arXiv:0806.1218 [hep-th]].

[56] M. Gualtieri, *Branes on Poisson varieties*, arXiv:0710.2719 [math.DG].

[57] I. T. Ellwood, *NS-NS fluxes in Hitchin's generalized geometry*, JHEP **0712**, 084 (2007) [arXiv:hep-th/0612100].

[58] M. Graña, R. Minasian, M. Petrini, D. Waldram, *T-duality, Generalized Geometry and Non-Geometric Backgrounds*, JHEP **0904**, 075 (2009) [arXiv:0807.4527 [hep-th]].